

# A stable vacuum monopole condensate in QCD

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A stationary finite energy density monopole solution in a pure  $SU(3)$  quantum chromodynamics (QCD) is proposed. The solution describes a colored Wu-Yang monopole dressed in gluon field. We have proved that such a classical solution corresponds to a stable vacuum monopole condensate in quantum theory. The generation of a mass gap and QCD vacuum structure are discussed.

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The structure of non-perturbative QCD vacuum and confinement phenomenon represent two most important and closely related problems in foundations of QCD. Whereas the mechanism of quark confinement is now well understood, there is much less knowledge about the microscopic vacuum structure and origin of color confinement. One of the attractive mechanisms of quark confinement is based on the Meissner effect in dual color superconductor which assumes generation of the monopole condensation [1–4]. The old QCD vacuum based on homogeneous magnetic field configuration [5, 6] suffers from instability against the quantum fluctuations. Further numerous studies of the QCD vacuum show that vacuum quantum stability has been a most serious and long-standing problem towards consistent description of a true non-perturbative vacuum [7–12].

In the present Letter we consider a new classical stationary solution which can be treated as a Wu-Yang monopole dressed in gluon field. We prove that such a solution provides a stable vacuum monopole condensate. We start with a Lagrangian for a pure  $SU(3)$  QCD

$$\mathcal{L} = -\frac{1}{4}F_{a\mu\nu}F^{a\mu\nu}. \quad (1)$$

We consider the following spherically symmetric ansatz for non-vanishing components of the gauge potential in spherical coordinates  $(r, \theta, \varphi)$ :

$$\begin{aligned} A_\theta^2 &= \psi_1(t, r), & A_\theta^5 &= \psi_2(t, r), \\ A_\varphi^1 &= -\psi_1(t, r) \sin \theta, & A_\varphi^4 &= \psi_2(t, r) \sin \theta, \\ A_\varphi^3 &= \frac{1}{g} \cos \theta, & A_\varphi^8 &= -\frac{\sqrt{3}}{g} \cos \theta, \end{aligned} \quad (2)$$

where Abelian gauge potentials  $A_\varphi^3$  and  $A_\varphi^8$  describe a static Wu-Yang monopole with a total color magnetic charge two,  $g_m^{\text{tot}} = 2$ , [13], and the functions  $\psi_1(t, r)$  and  $\psi_2(t, r)$  correspond to dynamic degrees of freedom of the gluon field. Substitution of the ansatz (2) into the equations of motion of the pure  $SU(3)$  QCD results in two

non-trivial partial differential equations

$$\begin{aligned} \partial_t^2 \psi_1 - \partial_r^2 \psi_1 + \frac{g^2}{2r^2} \psi_1 \left( 2\psi_1^2 - \psi_2^2 - \frac{2}{g^2} \right) &= 0, \\ \partial_t^2 \psi_2 - \partial_r^2 \psi_2 + \frac{g^2}{2r^2} \psi_2 \left( 2\psi_2^2 - \psi_1^2 - \frac{2}{g^2} \right) &= 0. \end{aligned} \quad (3)$$

In a special case when  $A_\theta^5 = A_\varphi^4 = A_\varphi^8 = 0$  the ansatz leads to a  $SU(2)$  embedded field configuration which corresponds to a system of a Wu-Yang monopole with a unit monopole charge,  $g_m^{\text{tot}} = 1$ , interacting to the off-diagonal gluon. With  $\psi_1(t, r) \equiv \psi(t, r)$ ,  $\psi_2(t, r) = 0$  the equations (3) reduce to one differential equation

$$\partial_t^2 \psi - \partial_r^2 \psi + \frac{g^2}{r^2} \psi \left( \psi^2 - \frac{1}{g^2} \right) = 0. \quad (4)$$

For simplicity we consider first the vacuum stability problem in the case of  $SU(2)$  embedded solution. For arbitrary function  $\psi(t, r)$  one has the following non-vanishing field strength components

$$\begin{aligned} F_{t\varphi}^1 &= -\partial_t \psi \sin \theta, & F_{r\varphi}^1 &= -\partial_r \psi \sin \theta, \\ F_{t\theta}^2 &= \partial_t \psi, & F_{r\theta}^2 &= \partial_r \psi, & F_{\theta\varphi}^3 &= \frac{1}{g} (g^2 \psi^2 - 1) \sin \theta. \end{aligned} \quad (5)$$

The energy functional is simplified as

$$E = 4\pi \int dr \left( (\partial_t \psi)^2 + (\partial_r \psi)^2 + \frac{1}{2g^2 r^2} (g^2 \psi^2 - 1)^2 \right). \quad (6)$$

The potential term in the energy functional is represented precisely by the radial magnetic field component  $F_{\theta\varphi}^3$  which generates a non-zero magnetic flux through a sphere with a center at the origin,  $r = 0$ . So that, the color magnetic charge of the monopole depends on time and distance from the center. Note that various generalized static Wu-Yang monopoles have been considered before (see [14] and refs. therein), all of them have singularities in agreement with the Derrick's theorem. The equation (4) admits a local non-static solution near the origin

$$\psi(t, r) = \frac{1}{g} + c_2(t)r^2 + \mathcal{O}(r^{n \geq 4}), \quad (7)$$

which removes the singularity of the monopole at the center and provides the finite energy density. In asymptotic region,  $r \simeq \infty$ , the non-linear equation (4) reduces to a free D'Alembert equation which has a standing spherical wave solution

$$\psi(t, r) \simeq a_0 + a_1 \cos(Mr) \sin(Mt) + \mathcal{O}\left(\frac{1}{r}\right), \quad (8)$$

where  $a_0, a_1$  are integration constants, the mass scale  $M$  appears due to scaling invariance in the theory. To solve numerically the Eq. (4) we use the local solution (7) as an initial condition along the boundary  $r = L_0$ . We choose the function  $c_2(t)$  in the form of a Jacobi elliptic function,  $c_2(t) = c_0 + c_1 \text{sn}\left[\frac{T_0}{2\pi}t, -1\right]$  ( $T_0 \equiv 4K[-1] \simeq 5.244 \dots$ ), since in the strong coupling regime,  $g \gg 1$ , the local solution near the origin in the leading order is given exactly by  $\text{sn}[t, -1]$ . With this one can solve numerically the equation (4), and the corresponding solution is depicted in Fig.1. The obtained solution has several

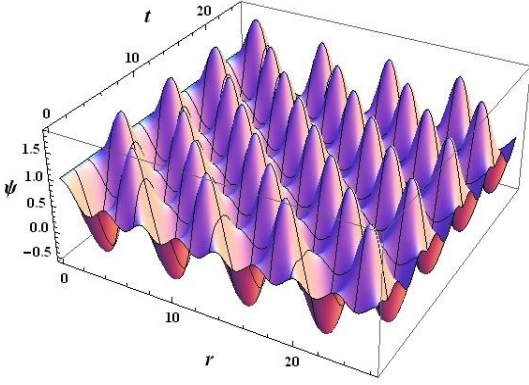


FIG. 1: Stationary monopole solution; ( $0 \leq r, t \leq L$ ),  $L = 8\pi$ ,  $c_0 = -0.0415$ ,  $c_1 = -0.56$ ,  $g = 1$ .

surprising features. The field configuration of the solution includes a static Wu-Yang monopole immersed in the standing spherical wave made of off-diagonal gluons. Another feature of the solution is that the corresponding canonical spin density vanishes identically. This gives a hint that such a solution, treated as a quantum mechanical wave function, can describe a stable condensate of massive spinless particles.

In order to prove the stability of the monopole condensate we study the structure of the effective action of QCD using the standard functional integral methods. An initial gauge potential  $A_\mu^a$  is split into a classical  $\mathcal{B}_\mu^a$ , and a quantum  $Q_\mu^a$  parts

$$A_\mu^a = \mathcal{B}_\mu^a + Q_\mu^a. \quad (9)$$

We choose a temporal gauge for the classical and the quantum field,  $\mathcal{B}_t^a = Q_t^a = 0$ . The temporal gauge has a residual symmetry which can be fixed by imposing an additional covariant Coulomb gauge condition,  $\mathcal{D}_i Q^{ia} = 0$ , with a covariant derivative  $\mathcal{D}_i$  containing the background

field  $\mathcal{B}_i^a$ . Hereafter we use the Latin indices for the spacial components. A one-loop correction to the classical action is obtained by functional integration over the quantum fields  $Q_i^a$ , and it can be expressed in terms of functional logarithms in a compact form [15–21]

$$\begin{aligned} S_{\text{eff}}^{\text{loop}} &= -\frac{1}{2} \text{Tr} \ln[K_{ij}^{ab}] + \text{Tr} \ln[M_{\text{FP}}^{ab}], \\ K_{ij}^{ab} &= -\delta^{ab} \delta_{ij} \partial_t^2 - \delta_{ij} (\mathcal{D}_k \mathcal{D}^k)^{ab} - 2f^{acb} \mathcal{F}_{ijc}, \\ M_{\text{FP}}^{ab} &= -(\mathcal{D}_k \mathcal{D}^k)^{ab}, \end{aligned} \quad (10)$$

where the operators  $K_{ij}^{ab}$  and  $M_{\text{FP}}^{ab}$  correspond to the contributions of gluons and Faddeev-Popov ghosts, respectively. The operator  $K_{ij}^{ab}$  represents a well-defined second order differential operator of elliptic type in four-dimensional Euclidean space  $(t, r, \theta, \varphi)$ . The question of vacuum stability is reduced to the question of whether the operator  $K_{ij}^{ab}$  has negative eigenvalues for a given classical background field  $\mathcal{B}_i^a$ . To find the eigenvalues one has to solve the Schrödinger type equation with the operator  $K_{ij}^{ab}$  treated as a Hamiltonian operator for a quantum mechanical system

$$K_{ij}^{ab} \Psi_j^b = \lambda \Psi_i^a, \quad (11)$$

where the “wave functions”  $\Psi_i^a(t, r, \theta, \varphi)$  describe quantum gluon fluctuations. Note that the ghost operator originates from the interaction of spinless ghost fields with the color magnetic field. Such interaction does not produce negative tachyon modes [6], so it is sufficient to study the eigenvalue spectrum of the operator  $K_{ij}^{ab}$  only. The operator  $K_{ij}^{ab}$  does not have explicit dependence on azimuthal angle  $\varphi$ , so that the ground state eigenfunctions  $\Psi_i^a$  depend only on three coordinates  $(t, r, \theta)$ . Before solving the Schrödinger equation (11) numerically, let us perform a qualitative estimate of ground state eigenvalues to trace the origin of positiveness of the operator  $K_{ij}^{ab}$ . First, we apply the variational method to reduce the three-dimensional equation (11) to an effective equation in two-dimensional space-time  $(t, r)$ . Within the variational approach one has to minimize the following “Hamiltonian” functional

$$\langle \mathcal{H} \rangle = \int dr d\theta d\varphi r^2 \sin \theta \Psi_i^a K_{ij}^{ab} \Psi_j^b. \quad (12)$$

One can make qualitative estimates assuming that all ground state eigenfunctions  $\Psi_i^a$  includes angle dependence which guarantee the finiteness of the Hamiltonian

$$\Psi_i^a(t, r, \theta, \varphi) = f_i^a(t, r) \sin \theta. \quad (13)$$

With this one can perform integration over the angle variables  $(\theta, \varphi)$  in (12) and obtain an effective Schrödinger equation for the ground state

$$\tilde{K}_{ij}^{ab} f_j^b(t, r) = \lambda f_i^a(t, r), \quad (14)$$

where the operator  $\tilde{K}_{ij}^{ab}$  includes dependence only on two coordinates  $(t, r)$ ,

$$\tilde{K}_{ij}^{ab} = \delta_{ij} \delta^{ab} \left( -\partial_t^2 - \partial_r^2 - \frac{2}{r} \partial_r \right) + V_{ij}^{ab}(t, r). \quad (15)$$

The quadratic form corresponding to the potential  $V_{ij}^{ab}$  can be written in the form

$$f_i^a V_{ij}^{ab} f_j^b = \frac{1}{r^2} V_0[f] + \frac{1}{r^2} V_1[f] + \frac{\psi^2}{r^2} V_2[f] + \frac{\psi}{r^2} V_3[f] + \frac{\partial_r \psi}{r} V_4[f], \quad (16)$$

where the first term includes contribution from a free vector Laplace operator,  $V_1[f]$  corresponds to contribution of the Wu-Yang monopole and  $V_{2,3,4}[f]$  contains contributions proportional to  $\psi^2$ ,  $\psi$  and  $\partial_r \psi$ , respectively. One can verify by using variational minimization procedure that a total coefficient function in front of the centrifugal potential  $1/r^2$  is positively defined for arbitrary fluctuating functions  $f_i^a$ . So that the effective Schrödinger equation (14) contains a positive centrifugal potential and a Coulomb type potential with the oscillating coefficient function  $\partial_r \psi V_4[f]$ . It is known that a quantum mechanical problem for a particle in the potential well with small enough depth does not admit bound states (in the space of dimension  $d \geq 3$ ). Therefore, there should be a critical value of the amplitude  $a_1$  of the monopole solution, (8), below which the eigenvalue spectrum becomes positive. Indeed, numerical study of solutions to the Eq. (14) implies a critical value  $a_{1cr}^{bound} = 1.3$ . A typical field configuration of  $f_i^a$  with a background monopole solution field with the asymptotic parameters  $a_0 = 0.895$ ,  $a_1 = 0.615$  is depicted in Fig.2. The estimate provides only an up-

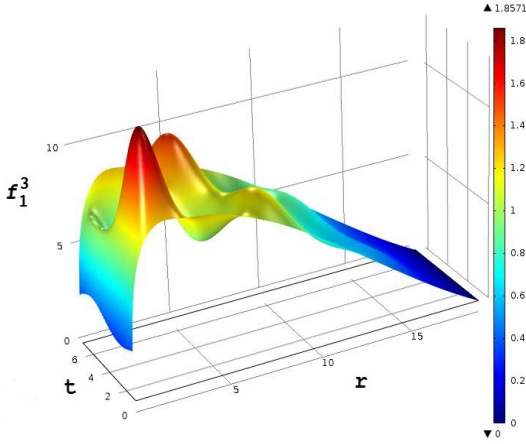


FIG. 2: Wave function  $f_1^3$  for the ground state with eigenvalue  $\lambda = 0.0293$ ,  $L = 6\pi$ .

per bound for the critical parameter  $a_{1cr}$ , so one should solve exactly the original eigenvalue equation (11). It turns out that exact solutions corresponding to the lowest eigenvalue possess striking features: there are only three non-vanishing functions,  $\Psi_1^3$ ,  $\Psi_3^2$  and  $\Psi_2^1 = \Psi_3^2$ . A key property is that all non-vanishing functions have no dependence on angle  $\theta$ , Fig. 3. This implies, that on the space of ground state solutions the original equations (11) reduce to two differential equations on two-dimensional space-time. We have obtained that the lowest eigenvalue

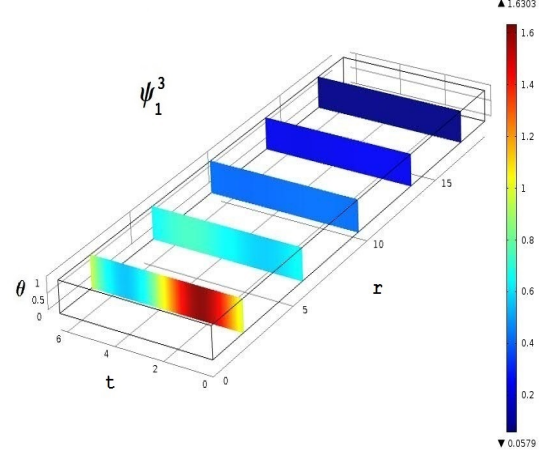


FIG. 3: Solution for the eigenfunction  $\tilde{\Psi}_1^3(t, r, \theta)$  with  $\lambda = 0.014218$ ; each slice represents a density plot.

is positive when asymptotic monopole amplitude is less than a critical value  $a_{1cr} \simeq 0.625$ .

In the case of  $SU(3)$  QCD the numeric analysis of ground state solutions of the full Schrodinger equation (11) with charge two monopole background leads to a similar factorization property. The ground state solution contains two sets of non-vanishing  $\theta$  independent functions,  $\tilde{\Psi}_1^3, \tilde{\Psi}_3^2, \tilde{\Psi}_2^1 = \tilde{\Psi}_3^2$  and  $\tilde{\Psi}_1^8, \tilde{\Psi}_7^3, \tilde{\Psi}_2^6 = \tilde{\Psi}_3^2$ . The functions obey the constraints,  $\tilde{\Psi}_7^3 = \tilde{\Psi}_3^2$  and  $\tilde{\Psi}_1^8 = \sqrt{3}\tilde{\Psi}_1^3$ . Due to this one has the following reduction of the original equations (11)

$$\begin{aligned} \partial_r^2 \tilde{\Psi}_1^3 + \frac{2}{r} \partial_r \tilde{\Psi}_1^3 + \partial_t^2 \tilde{\Psi}_1^3 \\ - \frac{1}{r^2} \left( (2\tilde{\psi}^2 + 1) \tilde{\Psi}_1^3 - 2(\tilde{\psi} - r\tilde{\psi}_r) \tilde{\Psi}_3^2 \right) &= 0, \\ \partial_r^2 \tilde{\Psi}_3^2 + \frac{2}{r} \partial_r \tilde{\Psi}_3^2 + \partial_t^2 \tilde{\Psi}_3^2 \\ - \frac{1}{2r^2} \left( (3\tilde{\psi}^2 - 2) \tilde{\Psi}_3^2 - 4(\tilde{\psi} - r\tilde{\psi}_r) \tilde{\Psi}_1^3 \right) &= 0. \end{aligned} \quad (17)$$

After simple rescaling of the monopole solution  $\tilde{\psi} = \sqrt{2}\psi$  and function  $\tilde{\Psi}_3^2 = \sqrt{2}\Psi_3^2$  ( $\tilde{\Psi}_1^3 = \Psi_1^3$ ) the Eqs. (17) reproduce the equations obtained due to reduction in the case of  $SU(2)$  QCD. The exact solution profile for the function  $\tilde{\Psi}_1^3$  is shown in Fig.4. We have verified that with increasing the space interval  $L \rightarrow \infty$  the corresponding eigenvalue  $\lambda(L)$  vanishes asymptotically from positive values. The eigenvalue dependencies  $\lambda(L)$  obtained from solving the exact, (11), and approximate, (14), Schrödinger equations are presented in Fig.5. Note that in  $SU(3)$  QCD one has a stable monopole condensate for both stationary monopole solutions, for the embedded  $SU(2)$  monopole and for  $SU(3)$  monopole with magnetic charge two as well. The  $SU(3)$  symmetrical monopole solution is preferable since for constant valued functions  $\psi_1, \psi_2$  the corresponding classical potential has an absolute minimum at  $\psi_1 = \psi_2 = \sqrt{2}$ . This is similar to the behavior of one-loop effective potential for constant

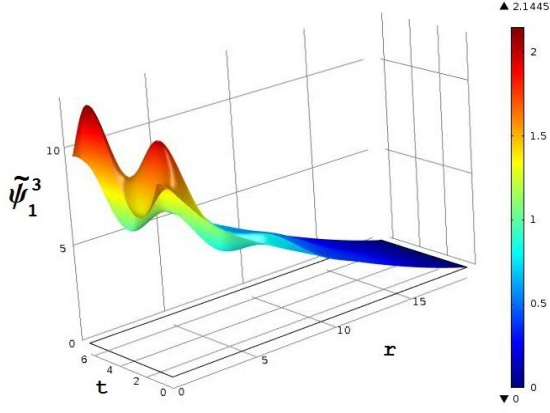


FIG. 4: Wave function  $\tilde{\Psi}_1^3$  for the ground state with eigenvalue  $\lambda = 0.02038$ ,  $L = 6\pi$ ,  $a_0 = 1.312$ ,  $a_1 = 0.615$ .

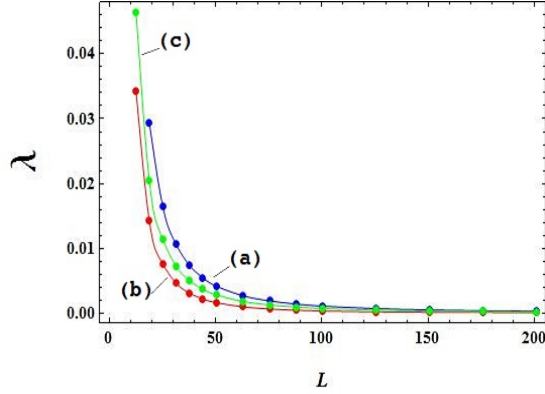


FIG. 5:  $\lambda$ -dependencies on the size  $L$  of the numeric domain: (a) approximate  $SU(2)$  solution, (b) exact numeric  $SU(2)$ , (c) exact numeric  $SU(3)$  solution.

magnetic fields where the potential has an absolute minimum for non-vanishing values of both magnetic fields  $H_3$  and  $H_8$  corresponding to Cartan subalgebra of  $\mathfrak{su}(3)$  [22].

The quantum gluon dynamics leads to generation of a

mass gap which fixes the value of the vacuum monopole condensate. In one-loop approximation the effective potential of a pure  $SU(N)$  QCD is well known [5, 15–21]

$$V_{\text{eff}} = \frac{1}{4}H^2 + \frac{11Ng^2(\mu)}{48\pi^2}H^2 \left( \ln \frac{g(\mu)}{\mu^2} - c \right), \quad (18)$$

where  $g(\mu)$  is a renormalized coupling constant defined at the subtraction point  $\mu^2 \simeq \Lambda_{\text{QCD}}$  and  $H^2 \equiv \vec{F}_{\mu\nu}^2$  is a color magnetic field. The potential has a non-trivial minimum at non-zero value for the vacuum monopole condensate  $\langle H \rangle \simeq 0.14\mu^2$ . The value of the monopole condensate determines the mass gap in QCD which is an order parameter in the theory. The mass scale parameter  $M$  characterizes the microscopic scale of the vacuum monopole condensate. In the confinement phase one assumes that the periodic structure of the classical monopole solution is defined by the parameter  $2\pi/M$  which should be less than inverse deconfinement temperature parameter  $kT_{\text{dec}}$ . This implies a lower bound for the mass scale,  $M \geq 2\pi kT_{\text{dec}}$ , which is of the same order as  $\Lambda_{\text{QCD}}$ . With increasing temperature, when the scale of the monopole configuration becomes comparable with the deconfinement scale one cannot apply averaging over the minimal space-time domain defined by the mass scale  $M$ . So that the vacuum averaging value  $\langle 0|A_\mu^a|0 \rangle$  becomes non-vanishing, which corresponds to the deconfinement phase with spontaneous symmetry breaking and gluon can be observed as a color object.

Our result on a stable one monopole condensation gives a hope that a stable condensate of two monopoles should exist as well. We expect that such monopole condensation can be realized in QCD in full analogy with Cooper electron pair condensation in the superconductor, as it was conjectured in the seminal papers [1–4].

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